Integration by differentiating with respect to a parameter

Below we illustrate how one computes certain definite integrals by differentiating under the integral. The trick was one of the favorites of the famous physicist Richard Feynman, who popularized the trick (although the trick dates back to well before him). At the level of our course, we do not have enough tools to justify the computations given below. However, the trick is beautiful, and the mathematics major should learn to justify the trick rigorously at some point.

1. Show that

$$\int_0^{2\pi} \ln |a - e^{i\theta}| d\theta = 0.$$

(Hint: First show that it suffices to show that

$$\int_0^{2\pi} \ln(a^2 + 2a\cos\theta + 1) d\theta = 0.$$

Next, the left hand side is a function of a. It is clearly zero when $a = 0$. Now differentiate the left hand side with respect to a. We would be done if we can show that the derivative with respect to a is zero, i.e.

$$\int_0^{2\pi} \frac{a + \cos\theta}{a^2 + 2a\cos\theta + 1} d\theta = 0.$$

But this last integral can be calculated using t-substitution: $t = \tan(\theta/2)$... Actually the original integral is best evaluated using complex analysis.)

2. Evaluate the improper integral

$$\int_0^1 \frac{x^2 - 1}{\ln x} dx.$$

(Hint: Let $I(t) = \int_0^1 \frac{x^t - 1}{\ln x} dx$. Then

$$I'(t) = \int_0^1 \frac{\partial}{\partial t} \frac{x^t - 1}{\ln x} dx = \int_0^1 x^t dx = \frac{1}{t+1}.$$

Hence $I(t) = I(0) + \ln(t+1) = \ln(t+1)$. The desired integral is $I(2) = \ln 3$.)

3. Evaluate the improper integral

$$\int_0^\infty \frac{\sin x}{x} dx.$$

(Hint: Let $I(t) = \int_0^\infty \frac{\sin x}{x} e^{-tx} dx$. Then

$$I'(t) = \int_0^\infty \sin x \, e^{-tx} dx = -\frac{1}{1+t^2}.$$

So $I(t) = C - \arctan t$ for some constant C. Letting $t \to +\infty$, we see that $I(t) \to 0$, so $C = \frac{\pi}{2}$, and the desired integral is $I(0) = \frac{\pi}{2}$. Again the original integral can be evaluated using complex analysis.)

4. Evaluate the improper integral

$$\int_0^\infty e^{-x^2} dx.$$

Hence, or otherwise, show that

$$\int_{-\infty}^\infty e^{-\pi x^2} dx = 1.$$

(Hint: To evaluate $A := \int_0^\infty e^{-x^2} dx$, let $I(t) = \int_0^\infty \frac{e^{-x^2}}{1+(x/t)^2} dx$. Then $A = \lim_{t\to+\infty} I(t)$. But

$$I(t) = t \int_0^\infty \frac{e^{-t^2 y^2}}{1+y^2} dy$$

after a change of variable. Hence

$$\frac{d}{dt}\left(\frac{e^{-t^2} I(t)}{t}\right) = -2t \int_0^\infty e^{-t^2(1+y^2)} dy = -2e^{-t^2} A.$$

Integrating both sides in t from 0 to $+\infty$, we get

$$\lim_{t\to+\infty} \frac{e^{-t^2} I(t)}{t} - \lim_{t\to 0^+} \frac{e^{-t^2} I(t)}{t} = -2A^2.$$

The first limit on the left hand side is 0 since $\lim_{t\to+\infty} \frac{e^{-t^2}}{t} = 0$ and $\lim_{t\to+\infty} I(t) = A$ is finite. The second limit on the left hand side can be computed using L'Hopital's rule, since $\lim_{t\to 0^+} I(t) = 0$. Indeed,

$$\lim_{t\to 0^+} \frac{e^{-t^2} I(t)}{t} = \lim_{t\to 0^+} [e^{-t^2} I'(t) - 2te^{-t^2} I(t)] = \lim_{t\to 0^+} I'(t) = \lim_{t\to 0^+} \int_0^\infty \frac{e^{-t^2 y^2}}{1+y^2} dy = \int_0^\infty \frac{1}{1+y^2} dy = \frac{\pi}{2}.$$

Hence $2A^2 = \pi/2$, i.e. $A = \sqrt{\pi}/2$. Since e^{-x^2} is even, this shows $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$, from which it follows that $\int_{-\infty}^\infty e^{-\pi x^2} dx = 1$. This last identity allows one to define the normal distribution as one whose probability density function is $e^{-\pi x^2}$.)